Stability Analysis of the Simplest Takagi-Sugeno
Fuzzy Control System Using Popov Criterion

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Abstract — In our paper, the properties of the simplest Takagi-Sugeno (T-S) fuzzy controller are first investigated. Next, based on the well-known Popov criterion with graphical interpretation, a sufficient condition in the frequency domain is proposed to guarantee the globally asymptotical stability of the simplest T-S fuzzy control system. Since this sufficient condition is presented in the frequency domain, it is of great significance in designing the simplest T-S fuzzy controller in the frequency domain.

Keywords: Takagi-Sugeno (T-S) fuzzy controllers, Popov criterion, stability analysis, frequency response methods.

1. Introduction

The Takagi-Sugeno (T-S) fuzzy model [1] is a landmark in the history of fuzzy control theory. Numerous fuzzy control problems, such as stability analysis, systematic design, robustness, and optimality, can be addressed under the framework of this T-S model [2]. Especially, given a T-S fuzzy model, a fuzzy controller design method named Parallel Distributed Compensation (PDC), has been proposed by Sugeno and Kang [3]. The corresponding stability analysis is also discussed in one of their papers [4]. The unique advantage of the PDC technique is that a lot of conventional linear controller design solutions based on both classical and modern control theory, which are actually for linear control systems, can be deployed in designing the nonlinear T-S fuzzy controllers as well.
As we know, frequency response methods have been well-developed, and widely used in industrial applications, which are straightforward and easy to follow by practicing engineers. The negative effect of noise in a control system can be evaluated by its frequency response. This advantage is very useful for control system analysis, since unavoidable noise usually deteriorates the overall control performance [5]. Besides, two popular frequency response methods, Bode and Nyquist plots, can provide a graphic insight into the control systems under study, and help engineers synthesize the corresponding controllers. Therefore, fusion of the T-S fuzzy model and frequency response methods is emerging in the field of control engineering. It is apparently necessary to analyze the stability of T-S fuzzy control systems in the frequency domain, when the frequency response methods are utilized in designing T-S fuzzy controllers.

The frequency response methods have been employed in both the Mamdani and T-S fuzzy control systems. In [6], the describing function approach is used to analyze the stability of a Mamdani fuzzy control system. Various types of Mamdani fuzzy controllers, e.g., Single-Input-Single-Output (SISO) and Multiple-Input-Multiple-Output (MIMO), are further investigated based on this describing function method in [7]-[10]. [11] and [12] discuss the application of the circle criterion and its graphical interpretation in the stability analysis of the SISO and MIMO Mamdani fuzzy control systems. In [13]-[15], the stability of the Mamdani fuzzy controllers is explored based on the Popov criterion. The describing function method is also used in the T-S fuzzy control systems in [16]. The multi-variable circle criterion is utilized to analyze the stability of the T-S fuzzy controllers in [16] and [17]. In this paper, we investigate the globally asymptotical stability of the simplest T-S fuzzy control system by using the famous Popov criterion with graphical interpretation in the frequency domain.

Our paper is organized as follows: in Section II, the simplest T-S fuzzy control system is first discussed. In Section III, the principles of the Popov criterion are briefly introduced. Next, a sufficient condition is derived to guarantee the globally asymptotical stability of the simplest T-S fuzzy control system in Section IV. A numerical example is presented in Section V to demonstrate how to employ this condition for the stability analysis of the simplest T-S fuzzy controller. Finally, some conclusions are drawn in Section VI.

2. Configuration of the simplest T-S fuzzy control system

The structure of the simplest T-S fuzzy control system to be explored in
our paper is shown in Fig. 1, where FLC and \( G(s) \) are the T-S fuzzy controller and plant to be controlled, respectively, \( r \) is the reference input, \( e \) is the feedback error, \( u \) is the controller output, and \( y \) is the system output.

![Fig. 1. Structure of the simplest T-S fuzzy control system.](image)

This simplest T-S fuzzy controller can be described by the following two rules:

If \( e \) is \( A \), then \( u_1 = k_1 e \),

If \( e \) is \( B \), then \( u_2 = k_2 e \),

where \( e \) is the input of this T-S fuzzy controller, \( u_i, i=1,2 \) are the outputs of the local consequent controllers, which are both proportional controllers here. It should be pointed out that \( k_i, i=1,2 \), gains of these local controllers, are assumed to be positive in this paper. Both \( A \) and \( B \) are fuzzy sets, and we use the triangular membership functions to quantify them, as shown in Fig. 2.

![Fig. 2. Membership functions of \( A \) and \( B \).](image)

\( A \) and \( B \) can be written as follows:

\[
\mu_A(e) = \begin{cases} 
0, & e < -a \\
\frac{1}{a}(e + a), & -a \leq e < 0 \\
\frac{1}{a}(e - a), & 0 \leq e < a \\
0, & e \geq a 
\end{cases}
\]
3. Popov criterion

In this section, the Popov criterion [18] is briefly discussed, which is employed for the stability analysis of our simplest T-S fuzzy control system.

![Fig. 3. Structure of nonlinear system for Popov criterion.](image)

The structure of the nonlinear system for the Popov criterion is illustrated in Fig. 3, and it can be formulated by the following state equations:

\[
\begin{align*}
\dot{X} &= AX + bu, \\
\dot{\xi} &= u, \\
y &= cX + d\xi, \\
u &= -\Phi(y),
\end{align*}
\]

where \(X \in \mathbb{R}^n, \xi, u, y\) are all scalars, and \(A, b, c, d\) have commensurate dimensions. The nonlinear element \(\Phi: \mathbb{R} \to \mathbb{R}\) is a time-invariant nonlinearity belonging to sector \((0, k)\), where \(k > 0\) is a finite number.

Here, function \(\Phi: \mathbb{R} \to \mathbb{R}\) belongs to sector \((0, k)\), if both the following two assumptions are met:

(i). \(\Phi(0) = 0\),

(ii). \(\Phi(y)(ky - \Phi(y)) > 0, \forall y \neq 0\).

According to the state equations of the above nonlinear system, the transfer function of the linear system in the forward path is
\[ h(s) = \frac{d}{s} + c(sI - A)^{-1}b. \]  

**Popov Criterion:** Consider the above system, and suppose (i) the matrix \( A \) is Hurwitz, (ii) the pair \((A, b)\) is controllable, (iii) the pair \((c, A)\) is observable, (iv) \( d>0 \), and the nonlinear element \( \Phi \) belongs to sector \((0, k), 0 \), where \( k>0 \) is a finite number. Under these conditions, this system is globally asymptotically stable, if there exists a number \( r>0 \), such that

\[
\inf_{\omega \in \mathbb{R}} \Re[\arg(h(j\omega))] + \frac{1}{k} > 0.
\]

The graphical interpretation of the Popov criterion can be given as follows: suppose we plot \( \omega \Im h(j\omega) \) vs. \( \Re h(j\omega) \), when \( \omega \) varies from 0 to \( \infty \), which is known as the Popov plot of \( h(s) \), the nonlinear system is globally asymptotically stable if there exists a nonnegative number \( r \), such that the Popov plot of \( h(s) \) lies to the right of a straight line passing through the point \((-1/k,0)\) with a slope of \( 1/r \).

*Proof.* Refer to [18] for the proof of this theorem.

### 4. Analysis of the Simplest Takagi-Sugeno Fuzzy Control System

In this section, the Popov criterion is employed to analyze the stability of the above simplest T-S fuzzy control system. Several theorems and lemmas are proved to demonstrate that if certain hypotheses are satisfied, the stability of the nonlinear system illustrated in Fig. 1 can be analyzed by using the Popov criterion.

**Theorem 1:** Let \( \Phi(e) \) represent the functional mapping achieved by the simplest T-S fuzzy control system, the following equation holds:

\[
\Phi(-e) = -\Phi(e).
\]

*Proof.* The proof is omitted for the simplicity of our presentation.

In fact, the theorem states that the functional mapping achieved by this simplest T-S fuzzy controller is symmetric about the origin.

Based on Theorem 1, the nonlinear system described in Fig. 1 can be recast into the system represented in Fig. 4, in which the minus in front of variable \( e \) is taken from before the FLC to after the module. In other words, the two systems are equivalent based on Theorem 1. It is observed from Fig. 4, if the functional mapping achieved by the simplest fuzzy T-S controller belongs to some sector, and \( G(s) \) satisfies the assumptions in the Popov criterion, this theorem can be directly applied to the simplest T-S fuzzy control system.
Lemma 1: The following two statements are equivalent:

(i). \( G(y)(ky - \Phi(y)) > 0, \forall y \neq 0, \text{and}, \Phi(0) = 0, \) \hspace{1cm} (12)

(ii). \( 0 < y\Phi(y) < ky^2, \forall y \neq 0, \text{and}, \Phi(0) = 0, \) \hspace{1cm} (13)

where \( k \) is a positive number.

Proof. The proof is omitted here for convenience.

Theorem 2: Let \( \Phi(e) \) denote the functional mapping of the T-S fuzzy controller in Figs. 4 or 1, \( \Phi(e) \) belongs to sector \((0, k_2 + \varepsilon)\), where \( \varepsilon \) is a sufficiently small positive number, i.e., the following inequality holds:

(i). \( \Phi(0) = 0, \) \hspace{1cm} (14)

(ii). \( \Phi(e) = (k_2 + \varepsilon)e - \Phi(e)) > 0, \forall e \neq 0. \) \hspace{1cm} (15)

Proof.

The output of the simplest T-S fuzzy controller can be represented as:

\[
\Phi(e) = \frac{\mu_1(e)u_1 + \mu_2(e)u_2}{\mu_1(e) + \mu_2(e)} = \frac{\mu_1(e)k_1 + \mu_2(e)k_2}{\mu_1(e) + \mu_2(e)}e. \hspace{1cm} (16)
\]

Note that \( \mu_1(e) \geq 0, \) and \( \mu_2(e) \geq 0. \)

It is obvious that \( \Phi(e) = 0, \) if and only is \( e=0. \) Furthermore, there are

\[
k_1 \leq \frac{\mu_1(e)k_1 + \mu_2(e)k_2}{\mu_1(e) + \mu_2(e)} \leq k_2, \hspace{1cm} (17)
\]

\[
k_1e^2 \leq \mu_1(e)k_1 + \mu_2(e)k_2 \leq k_2e^2, \hspace{1cm} (18)
\]

\[
k_1e^2 \leq \Phi(e)e \leq k_2e^2. \hspace{1cm} (19)
\]

Obviously, there exists a sufficiently small positive number \( \varepsilon, \) such that \( 0 < \Phi(e) < (k_2 + \varepsilon)e^2, \) when \( e \) is not equal to zero. In view of Lemma 1, \( \Phi(e) \) belongs to sector \((0, k_2 + \varepsilon)\).

We can apply the Popov criterion to the stability analysis of our simplest T-S fuzzy control systems, as stated by the following theorem.

Theorem 3: The fuzzy control system shown in Fig. 1 is globally asymptotically stable, if the following set of conditions hold:

(i). \( G(s) \) can be represented by the state equations from (3) to (6),

(ii). the matrix \( A \) is Hurwitz,

(iii). the pair \( (A, b) \) is controllable, and the pair \( (c, A) \) is observable,
(iv). \( d > 0 \),
(v). there exists a number \( r > 0 \), such that
\[
\inf_{\omega \in \mathbb{R}} \text{Re} [(1 + j\omega h(j\omega))] + \frac{1}{k_2 + \varepsilon} > 0,
\]
where \( \varepsilon \) is a sufficiently small positive number.

Similar to the graphical interpretation of the Popov criterion, the graphical interpretation of the above theorem is that suppose we plot \( \text{Re} h(j\omega) \) vs. \( \omega \) \( \text{Im} h(j\omega) \), as \( \omega \) varies from 0 to \( \infty \), the equilibrium of our simplest T-S fuzzy system is globally asymptotically stable, if there exists a nonnegative number \( r \), such that the Popov plot of \( h(s) \) lies to the right of a straight line passing through the point \( \left( -\frac{1}{k_2 + \varepsilon}, 0 \right) \) with a slope of \( 1/r \).

As a matter of fact, if there exists a nonnegative number \( r \), such that the Popov plot of \( h(s) \) lies to the right of a straight line passing through point \( \left( -\frac{1}{k_2}, 0 \right) \), it is guaranteed that there is a line passing through point \( \left( -\frac{1}{k_2 + \varepsilon}, 0 \right) \), such that the Popov plot of \( h(s) \) lies to the right of this straight line. The discussion can be formally stated as the following corollary.

**Corollary 1:** The fuzzy control system shown in Fig. 1 is globally asymptotically stable, if the following set of conditions hold:
(i) - (iv) are the same with those in Theorem 3.
(v). there exists a number \( r > 0 \), such that
\[
\inf_{\omega \in \mathbb{R}} \text{Re} [(1 + j\omega h(j\omega))] + \frac{1}{k_2} > 0.
\]

In the next section, a numerical example is presented to demonstrate how to employ Theorem 3 or Corollary 1 in analyzing the stability of our T-S fuzzy control system.

5. Simulations

**Example.** In this example, a stable plant to be controlled is:
\[
G(s) = \frac{1}{s(s + 1)^2}.
\]

Two suitable proportional gains, \( k_1 = 0.2, k_2 = 0.5 \), are obtained based on the Bode plot of \( G(s) \). A simplest T-S fuzzy controller with the following two rules is constructed:

If \( e \) is \( A \), then \( u_1 = 0.2e \),
If \( e \) is \( B \), then \( u_2 = 0.5e \).

‘\( a \)’, a characteristics parameter of the input membership functions (refer to (1) and (2)), is \( \pi/8 \). Note, \( \pi/8 \) is chosen here only for convenience. In fact, \( a \) has no effect on the stability of the closed-loop fuzzy control system. In this example, the proportional compensators are designed only to show how to utilize Theorem 3 in the stability analysis of our fuzzy control system, which does not require the control performance of the system be perfect. Both the Popov plot of \( G(s) \) and a straight line passing through point \( \left( \frac{-1}{k_2}, 0 \right) \) with the slope of 0.5 are shown in Fig. 5. It is argued in Theorem 3 that the nonlinear system is globally asymptotically stable, if there exists a nonnegative number \( r \), such that the Popov plot of \( h(s) \) lies to the right of a straight line passing through point \( \left( \frac{-1}{k_2}, 0 \right) \) with a slope of \( 1/r \).

Hence, the T-S fuzzy control system in this example is globally asymptotically stable, since we can easily find such a straight line passing through point \( \left( \frac{-1}{k_2}, 0 \right) \), provided that \( k_2 \) is less than 2.

![Fig. 5. Popov plot of G(s) and a straight line (solid line represents Popov plot of G(s), and thin line represents straight line passing through point \( \left( \frac{-1}{k_2}, 0 \right) \) with a slope of 0.5).](image)

6. Conclusions

In our paper, the properties of the simplest T-S fuzzy controller are first investigated. A sufficient condition is next proposed to guarantee the glob-
ally asymptotically stability of the equilibrium point of this simplest T-S fuzzy control system by using the famous Popov criterion. The theorem derived based on the Popov theorem has a good graphical interpretation in the frequency domain. Thus, it can be employed in designing a T-S fuzzy controller in the frequency domain. Additionally, it can be observed that $a$, which is a characteristics parameter of the input membership functions, has no effect on the stability of the fuzzy control system. However, $a$ does affect the dynamical control performance. With the decrease and growth of $a$, the simplest T-S fuzzy controller converts to $u = k_2e$ and $u = k_1e$, respectively. Therefore, how to choose appropriate $a$ for achieving the optimal control performance is a challenging topic. We also emphasize that although only two fuzzy rules are examined here, the proposed stability analysis method is still applicable to the single-input T-S fuzzy controllers with multiple rules. Although the theorems derived in this paper have a graphical interpretation, their conditions imposed on the controlled plant are stringent, and the T-S fuzzy controller that has been analyzed is simple. In the future research, we are going to explore new stability theorems with graphical interpretation in the frequency domain for a wider class of plants as well as more complex T-S fuzzy controllers.

Acknowledgments

X. Z. Gao's research work was funded by the Academy of Finland under Grant 201353.

References


